Observe that G and $S_1, \dots, S_4$ must have an equal # of shaded and unshaded squares. So the same must be true for S_5. But, S_5 can never have the same #

1

$\S 0$ Combinatorics

Themes:

- ——
- ⋮
- power of labeling.

2

e.g.

Q!

Fsit

S_1

The number of permutations of list w/ n elements is $n!$

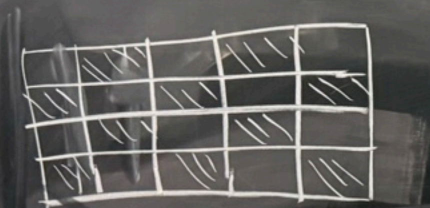
e.g.] a_1, a_2, a_3: 1) a_1, a_2, a_3, 2) a_2, a_1, a_3, 3) a_1, a_3, a_2, 4) a_3, a_2, a_1, 5) a_2, a_3, a_1, 6) a_3, a_1, a_2

the same #

2

§1 Tetris Stuff

Q]

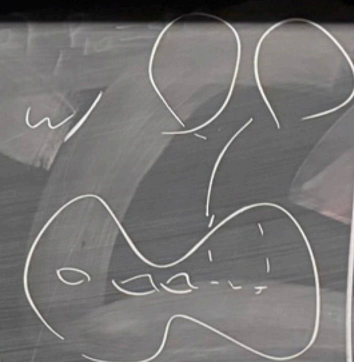
$G =$ 

Is it possible to tile G w/ the shapes

$= S_1$, $= S_2$, $= S_3$, $= S_4$, $= S_5$

3

Prop: It's impossible to tile G w/ the tetris blocks.



Pf: Consider the checkerboard shading. Observe that G and $S_1, \dots, S_4$ must have an equal # of shaded and unshaded squares. So the same must be true for S_5. But, S_5 can never have the same #.

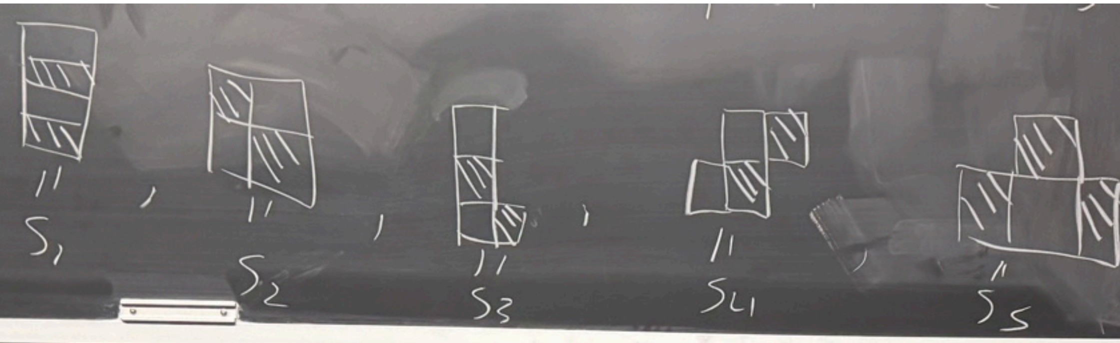
4 §2

Defn:

is a

Thm: T

e.g.] a, a



4 $\boxed{\text{§2 Permutations and Combinations}}$

<u>Defn</u>: A <u>permutation</u> of a list $a_1, \dots, a_n$ is a reordering $a_{m_1}, a_{m_2}, \dots, a_{m_n}$.

<u>Thm</u>: The number of permutations of list w/ n elements is $n!$

<u>e.g.</u> a_1, a_2, a_3: 1) a_1, a_2, a_3, 2) a_2, a_1, a_3, 3) a_1, a_3, a_2
4) a_3, a_2, a_1, 5) a_2, a_3, a_1, 6) a_3, a_1, a_2

same #

e.g.) $\binom{n}{1} = n$, $\binom{n}{k} = 0$ if $n < k$

Convent

5

Pf: We proceed by induction.

Base case: $n = 1$, one possible ordering a_1

So # of perms is $1 = 1!$.

IH: Assume that the # of perms of list w/ n elements is $n!$

Let $a_1, \dots, a_{n+1}$ be a list of $n+1$ elements and let $a_{i_1}, \dots, a_{i_{n+1}}$ be a permutation

6

This res

induction

So,

there ar

give ri

to the

$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$

- Convention: $0! = 1$

6

$a_1, a_2, \ldots, a_n, \boxed{a_{n+1}}$

$\boxed{a_i = a_{n+1}}$

$a_{m_1}, a_{m_2}, \ldots, a_{m_n}, a_{m_{n+1}}$

a_1, a_2, a_3

$a_1, a_3, a_2 \quad a_1, a_2, a_3$

a_1, a_2

$\boxed{\begin{matrix} a_3, a_1, a_2 \\ a_1, a_3, a_2 \\ a_1, a_2, a_3 \end{matrix}}$

This results in a permutation of $a_1, \ldots, a_n$. So, by induction there are $n!$ such permutations.

So, for each such reordering of $a_1, \ldots, a_n$ there are $n+1$ reorderings of $a_1, \ldots, a_{n+1}$ that give rise to the reordering of $a_1, \ldots, a_n$. So total # of reorderings is $(n+1) \cdot n! = (n+1)!$ ▨

procedure.

But, observe that double-counting: namely any permutation of $a_{m_1}, \dots, a_{m_k}$ results in the same set

$\boxed{7}$ Defn: For positive integers n, k the number $\binom{n}{k}$ denotes the # of ways of choosing k elements from an n element.

e.g.) $\binom{n}{1} = n$, $\binom{n}{k} = 0$ if $n < k$

$\boxed{8}$

Con

Pf:

reord

$n!$ Perms of $a_1, \dots, a_n$

$k!(n-k)!$

Selection of k things from $\{a_1, \dots, a_n\}$

Prop: Let $n \geq k$, then

$$\binom{n}{k} = \frac{n!}{k! \cdot (n-k)!}$$

Convention: $0! = 1$

$a_1 \; a_2 \; a_3$

$a_2 a_1 a_3 \qquad a_1 a_2 a_3$

$(3)(4) \cdot 5$

Pf: Consider the following procedure: take a reordering $a_{\pi_1}, \dots, a_{\pi_n}$ of an n-element list $a_1, \dots, a_n$

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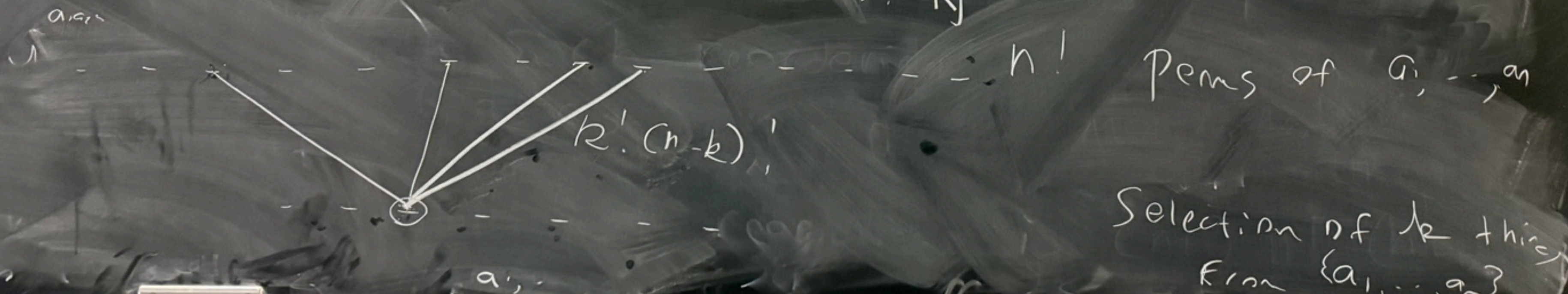
and consider the elements $a_{m_1}, \ldots, a_{m_k}$.

This procedure produces from a permutation of $a_1, \ldots, a_n$ a selection of k elements from $\{a_1, \ldots, a_n\}$. Clearly every such selection of k things from $\{a_1, \ldots, a_n\}$ arises from this procedure.

But, observe that double-counting; namely any permutation of $a_{m_1}, \ldots, a_{m_k}$ results in the same set procedure: take a reordering $a_{m_1}, \dots, a_{m_n}$ of an n-element list $a_1, \dots, a_n$

10

But, I can also reorder $a_{m_{k+1}}, \dots, a_{m_n}$.
This ambiguity accounts for all double-counting.
So for each selection of k elements from $\{a_1, \dots, a_n\}$ there are $k!(n-k)!$. The total # of selections is $\frac{n!}{k!(n-k)!}$ $\blacksquare$



Q: Is it possible to find a board state where none of the red boxes contain a stone?

$\boxed{11}$

Prop: $\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{k-r}$ $\quad n \geq k \geq r$

Pf: Note, the LHS counts # of ways to select k elements from an n element set, then r elements from that k element subset.

labeling squares containg a stone. (e.g. $W_0=1$)

Claim: $W_i = 1$ for all i.

Pf: By induction. $i=0$ is clear.

But RHS counts the number of ways of choosing an r-element set in $\{a_1, \ldots, a_n\}$ and then filling it out to a k-element set.

But, these are the same thing. $\boxed{}$

Pf: Note, the LHS counts # of ways to select k elements from an n element set, then r elements from that k element subset.

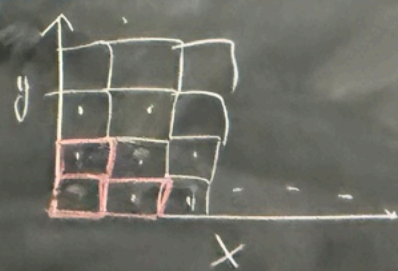
and then

But, these

13

§3 Escape!

Set up:



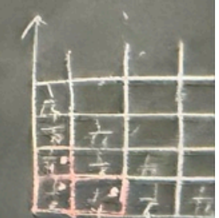
Admissible move

Q: Is it possible to find a board state where none of the red boxes contain a stone?

14

Prop: N

Pf.



For each

all numbers

Claim W

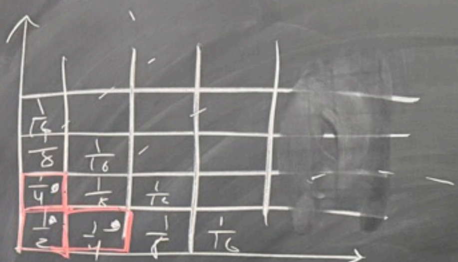
Pf. By ind

et, these are the same thing. ∎

14

Prop: No – the game is unwinnable.

Pf:



For each i (game step) W_i be the sum of all numbers labeling squares containing a stone. (e.g. $W_0 = 1$)

Claim: $W_i = 1$ for all i.

Pf: By induction. $i = 0$ is clear.

Q: Is it possible to find a board state where none of the red boxes contain a stone?

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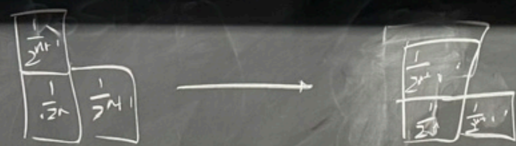
but  doesn't change W_i (!)

Assume that at some step i all the stones are not in the red boxes, this implies $W_i < \sum\begin{pmatrix}\text{labels in}\\ \text{non-red boxes}\end{pmatrix}$

$$\sum\begin{pmatrix}\text{All}\\ \text{boxes}\end{pmatrix} = \sum_{i=1}^{\infty} \sum\begin{pmatrix}\text{Row } i\\ \text{boxes}\end{pmatrix}$$

but $\sum\begin{pmatrix}\text{Row } i\\ \text{boxes}\end{pmatrix} = \sum_{n=i}^{\infty} \frac{1}{2^n} = \frac{1}{2^{i-1}}$

16

each (game step)
all numbers labeling squares containg a stone. (e.g. $W_0 = 1$)
Claim: $W_i = 1$ for all i.
Pf: By induction. $i=0$ is clear, but game

16

So $\sum \begin{pmatrix} \text{All} \\ \text{boxes} \end{pmatrix} = \sum_{i=1}^{\infty} \left(\sum \begin{pmatrix} \text{Row } i \\ \text{boxes} \end{pmatrix} \right)$

$$= \sum_{i=1} \frac{1}{2^{i-1}} = 2$$

$$\sum \begin{pmatrix} \text{Non-red} \\ \text{boxes} \end{pmatrix} = \sum \begin{pmatrix} \text{All} \\ \text{boxes} \end{pmatrix} - \sum \begin{pmatrix} \text{red} \\ \text{boxes} \end{pmatrix}$$

$$= 2 - 1$$

$$= 1$$

Contradiction.

$\boxtimes$